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ON GENUS AND CANCELLATION IN HOMOTOPY

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ABSTRACT

The *genus* is determined for spaces of the homotopy type of a CW complex with one cell each in dimensions 0, $2n$ and $4n$ (and no other cells), such spaces providing the only cases of spaces with two non-trivial cells such that the homotopy class of the attaching map for the top cell is of infinite order and the genus of the space is non-trivial. The genus is characterised completely by two well understood invariants: the *Hopf invariant* of the attaching map of the $4n$-cell and the genus of the *suspension* of the space. The algebraic tools are developed for the investigation of the $\vee$-*cancellation* behaviour of these spaces and a cancellation theorem is proved: the homotopy type of a finite wedge of such spaces determines the homotopy type of each of the summands as long as the attaching maps of the $4n$-cells all represent homotopy classes of infinite order. Comparing this result to known results about *finite co-H-spaces* shows that the Hopf invariant is the single obstruction to such spaces admitting a co-H structure.

One approach to an understanding of the homotopy theory of topological spaces is to try to decompose them with respect to the *one-point union* (or $\vee$-*product*) into spaces which are not themselves decomposable, in the hope that the homotopy properties of these indecomposable spaces are easier to deal with and that the homotopy properties of the original space can be recovered from them. For this programme to have any prospect of success, it would be necessary either to know that the $\vee$-decomposition of a given space is unique up to homotopy—and, of course, up to the order of the components—or, at the very least, to have control over the indeterminacy. Restricting attention to topological spaces of the homotopy type of CW-complexes of finite type (which we simply call *spaces*), the central questions are:

(i) *Given spaces A, B and C with $A \vee C$ and $B \vee C$ homotopically equivalent, what can be said about A and B?*

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(ii) *Given spaces A and B with $A \vee \cdots \vee A$ and $B \vee \cdots \vee B$ (with the same finite number of summands in each case) homotopically equivalent, what can be said about A and B?*

In general A and B need not be homotopically equivalent—that is to say we cannot always *cancel*. But in all the known examples of non-cancellation, the spaces A and B were almost homotopically equivalent in the sense that at each prime p their p-localisations $A_{(p)}$ and $B_{(p)}$ were homotopically equivalent. Spaces with this property are said to be *of the same genus*. The question then arises naturally: *Do genus and non-cancellation always go hand in hand?*

Indeed, they often do. Zabrodsky [Z] showed that if attention is restricted to *finite H-spaces*, then every non-trivial genus provides examples of non-cancellation phenomena with respect to cartesian product:

Proposition. *Let A and B be finite H-spaces of the same genus. Then there are a finite H-space C and a positive integer r such that* (i) *$A \times C$ and $B \times C$ are homotopically equivalent and* (ii) *A^r and B^r are homotopically equivalent. (Here A^r denotes cartesian product of r copies of A.)*

Wilkerson [CW] established a converse and claimed the corresponding result for *finite co-H-spaces* and $\vee$-product:

Proposition. *Let A and B be finite H-spaces (resp. co-H-spaces). Then the following are equivalent:*

(i) *A and B are of the same genus.*
(ii) *$A \times C$ and $B \times C$ (resp. $A \vee C$ and $B \vee C$) are homotopically equivalent, for some finite H-space (resp. co-H-space) C.*
(iii) *A^r and B^r are homotopically equivalent for some positive integer r, where A^r denotes the cartesian (resp. $\vee$-) product of r copies of A.*

In other words, genus and non-cancellation are inseparable in the cases of finite H-spaces and finite co-H-spaces.

What if the spaces are not required to be H-spaces or co-H-spaces? We investigate this question for finite simply connected CW-complexes with cells in dimensions 0, $2n$, and $4n$.

Theorem [M]. *The following statements are equivalent for spaces consisting of precisely three cells, one each in dimensions 0, n and m with $1 < n < m - 1$, as long as either the attaching map of the m-cell represents a suspension element in $\pi_{m-1}(S^n)$ or both the order in $\pi_{m-1}(S^n)$ of this attaching map and n are odd.*

(i) *The spaces are of the same genus.*
(ii) *The attaching maps of the top cells generate the same subgroup of* $\pi_{m-1}(S^n)$.
(iii) *The wedges of the spaces with* $S^n \vee S^m$ *are homotopically equivalent.*

This paper continues the investigation of spaces with three cells, but we now assume the attaching map of the cell in the top dimension to be of *infinite* order. The only infinite homotopy groups of spheres are $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$. Since the mapping cones of maps $S^n \to S^n$ $(n > 1)$ are simply connected Moore spaces, their homology groups determine them uniquely up to homotopy equivalence. Hence the only case left to consider is when the cells are arrayed in dimensions 0, $2n$ and $4n$ (with $n > 0$). This case provides a striking contrast to the theorem above, for while the genus can still be algebraically characterised in a manner which extends (ii), the geometrical condition (iii) no longer characterises the genus: in general (i) does not imply (iii) in the above theorem. Specifically,

THEOREM A. *Given spaces with precisely three cells, one each in dimensions* 0, $2n$ *and* $4n$, *they are of the same genus if and only if*
(i) *the attaching map of their* $4n$*-cells generate the same subgroup of* $\pi_{4n-1}(S^{2n})$ *modulo torsion and*
(ii) *the suspensions of these maps generate the same subgroup of* $\pi_{4n}(S^{2n+1})$.

We shall derive this theorem as a consequence of the characterisation of the genus of $S^{2n} \cup_f e^{4n} = C_f$ in terms of two invariants: the *genus of the suspension* and the *Hopf invariant of f*. (Observe that while it is obvious that the genus of the suspension is genus-invariant, it is not *immediately* obvious that the Hopf invariant is.)

It is worth noting that this theorem does not, in fact, require any restriction on the order of the attaching map of the cell in the top dimension, although, of course, our main interest is when this order is infinite.

THEOREM B. *Given spaces with precisely three cells, one each in dimensions* 0, $2n$ *and* $4n$, *such that the attaching maps of the* $4n$*-cells are of infinite order, they become homotopically equivalent by wedging with the wedge of a* $2n$*-sphere and a* $4n$*-sphere if and only if they are already homotopically equivalent themselves.*

This already appears in [M]. The proof given there applies a form of Hilton's "matrix calculus" [H2]. But the question is left open: *What can be said about two three-cell spaces knowing that the wedge of the first with itself is homotopically equivalent to the wedge of the second with itself?*

Our answer to this question is

THEOREM C. *Take two spaces with precisely three cells, one each in dimensions* 0, $2n$ *and* $4n$, *such that the attaching maps of the* $4n$-*cells are of infinite order. The wedge of finitely many copies of one is homotopically equivalent to the wedge of the same number of copies of the other if and only if the spaces are themselves homotopically equivalent.*

We derive Theorem B and Theorem C as corollaries to the following $\vee$-factorisation theorem, which we prove by extending and applying the "matrix calculus".

THEOREM D. *Take spaces with precisely three cells, one each in dimensions* 0, $2n$ *and* $4n$. *Two finite wedges of such spaces are homotopically equivalent if and only if the number of factors with the attaching map of the* $4n$-*cell of infinite order is the same in both wedges, these factors are pairwise homotopically equivalent and the wedges of the remaining factors are homotopically equivalent.*

The rest of the paper is devoted to the proofs of Theorem A (Corollary 2.4 below) and Theorem D (Theorem 4.13 below). Elements of Hopf invariant 1 arise only when $n = 1$, 2 or 4 [A] and occasionally create the need for slightly special arguments in those cases. The characterisation of the genus of C_f for $f \in \pi_{4n-1}(S^{2n})$ is illustrated by the explicit computation of the genera of the complex, the quaternionic and the Cayley projective planes.

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1. Homotopy properties of $S^{2n} \cup_f e^{4n}$

The CW-complexes we consider are of a particularly simple form: they are the mapping cones C_f of maps $f\colon \bigvee S^{4n-1} \to \bigvee S^{2n}$. By the Blakers–Massey Theorem [H1], maps between such mapping cones C_f and C_g arise from homotopy commutative diagrams

$$\begin{array}{ccc} \bigvee S^{4n-1} & \xrightarrow{\ f\ } & \bigvee S^{2n} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \bigvee S^{4n-1} & \xrightarrow[\ g\ ]{} & \bigvee S^{2n} \end{array}$$

We begin the algebraic analysis of such a commutative diagram — which we refer to as our "fundamental diagram" — by considering the special case in which each of the bouquet of spheres consists of a single sphere.

Then the horizontal maps are classified by $\pi_{4n-1}(S^{2n})$ and the vertical ones by

$\pi_{4n-1}(S^{4n-1})$ and $\pi_{2n}(S^{2n})$ respectively. These last groups operate on $\pi_{4n-1}(S^{2n})$ by the homotopy operation of *composition*. The other homotopy operations which play a rôle are *suspension* and the *Whitehead product*. We take the opportunity to summarise their properties and to fix notation.

The features of these operations essential for our investigations are that composition—which we denote by $\circ$—is always *right-additive* (additive in the second variable) but only sometimes *left-additive*, whereas the Whitehead product $[\ ,\ ]$: $\pi_t(X) \times \pi_u(X) \to \pi_{t+u-1}(X)$ is natural in maps $X \to Y$, bi-additive for $t,u \geq 2$, graded commutative and its image lies in the kernel of the suspension morphism—which we denote by Σ. The next theorem is a formal statement of these facts. Other, more special properties will be introduced as needed.

THEOREM 1.1 [GWW]. *Let $\alpha,\beta \in \pi_t(X)$, $\gamma,\delta \in \pi_u(X)$ and $\epsilon,\zeta \in \pi_r(S^t)$. Let η be a homotopy class of maps $X \to Y$. Then*

(i) $\alpha \circ (\epsilon + \zeta) = \alpha \circ \epsilon + \alpha \circ \zeta$.

(ii) *If, in addition, either X is an H-space or ϵ is a suspension element, then* $(\alpha + \beta) \circ \epsilon = \alpha \circ \epsilon + \beta \circ \epsilon$.

(iii) $\eta \circ [\alpha,\gamma] = [\eta \circ \alpha, \eta \circ \gamma]$.

(iv) $[\alpha + \beta, \gamma + \delta] = [\alpha,\gamma] + [\alpha,\delta] + [\beta,\gamma] + [\beta,\delta]$ *if* $t,u \geq 2$.

(v) $[\alpha,\gamma] = (-1)^{tu}[\gamma,\alpha]$.

(vi) $\Sigma[\alpha,\gamma] = 0$. ■

The Hilton-Milnor Formula describes the failure of the composition $\circ: \pi_t(X) \times \pi_r(S^t) \to \pi_r(X)$ to be left-additive.

THEOREM 1.2 [GWW] (Hilton-Milnor Formula). *Let $\alpha,\beta \in \pi_t(X)$ and $\epsilon \in \pi_r(S^t)$. Then*

$$(\alpha + \beta) \circ \epsilon = \alpha \circ \epsilon + \beta \circ \epsilon + \sum_{i \in \mathbf{N}} w_i(\alpha,\beta) \circ h_i(\epsilon)$$

where each $w_i(\alpha,\beta)$ is an iterated Whitehead product with at least two factors. If $w_i(\alpha,\beta)$ has $s+1$ factors, then h_i is a homomorphism $\pi_r(S^t) \to \pi_r(S^{(s+1)t-s})$ and is called the "i-th Hilton-Hopf invariant". ■

We are now ready to determine the operation of $\pi_{2n}(S^{2n})$ on $\pi_{4n-1}(S^{2n})$. Given $\alpha,\beta \in \pi_m(S^m)$ and $\gamma \in \pi_{2m-1}(S^m)$ $(m \geq 2)$, it follows from the Hilton-Milnor formula that

$$(\alpha + \beta) \circ \gamma = (\alpha \circ \gamma) + (\beta \circ \gamma) + [\alpha,\beta] \circ h_0(\gamma)$$

since the image of the Hilton-Hopf invariant h_i lies in $\pi_{2m-1}(S^{r_i})$ and $r_i \geq 2m$, so that $\pi_{2m-1}(S^{r_i}) = 0$ whenever $i > 0$. In fact, h_0 agrees with the classical *Hopf in-*

variant, which we denote by H and whose properties are summarised in the next theorem.

THEOREM 1.3 [GWW]. *$H: \pi_{2m-1}(S^m) \to \mathbf{Z}$ is a group homomorphism with the following properties. Take $\alpha \in \pi_{2m-1}(S^{2m})$, $\varphi \in \pi_m(S^m)$, ι the homotopy class of $\mathrm{id}_{S^m}: S^m \to S^m$ and $\psi \in \pi_{2m-1}(S^{2m-1})$. Then*

(i) $H(\varphi \circ \alpha) = (\deg \varphi)^2 H(\alpha)$.

(ii) $H(\alpha \circ \psi) = (\deg \psi) H(\alpha)$.

(iii) $\alpha \in \Sigma(\pi_{2m-2}(S^{m-1}))$ *if and only if* $H(\alpha) = 0$.

(iv) *If* $m = 2n - 1$, *then* $\mathrm{im}(H) = \{0\}$.

(v) *If* $m = 2n$, *then* $\mathrm{im}(H) = \begin{cases} \mathbf{Z} & \text{if } n \in \{1,2,4\}, \\ 2\mathbf{Z} & \text{otherwise.} \end{cases}$

(vi) $H([\iota, \iota]) = 2$. ∎

Only the homotopy groups $\pi_3(S^2)$, $\pi_7(S^4)$ and $\pi_{15}(S^8)$ contain elements of Hopf invariant 1. In each of these cases the corresponding Hopf map $\eta: S^{2m-1} \to S^m$ $(m = 2,4,8)$ represents one such homotopy class. These play an important rôle below.

THEOREM 1.4 [GWW].

$$\pi_{2m-1}(S^m) = \begin{cases} \mathbf{Z}\xi \oplus T_m & \text{if } m \text{ is even} \\ T_m & \text{if } m \text{ is odd} \end{cases}$$

where ξ is of infinite order and T_m, the torsion subgroup of $\pi_{2m-1}(S^m)$, is $\Sigma(\pi_{2m-2}(S^{m-1}))$. For $m \neq 2$, 4 or 8, ξ may be taken to be the Whitehead product $[\iota, \iota]$. Otherwise ξ may be taken to be the corresponding Hopf map η. Moreover $T_m \cong \mathbf{Z}/r\mathbf{Z}$ with $r = 1, 12, 120$ respectively, for $m = 2,4,8$. In these cases, T_m is generated by the suspension, ω, of a generator of $\pi_{4m-2}(S^{2m-1})$, and $[\iota, \iota] = 2\xi + \omega$. Finally, the suspension morphism $\Sigma: \pi_{4n-1}(S^{2n}) \to \pi_{4n}(S^{2n+1})$ is surjective with kernel generated by $[\iota, \iota]$, so that $\pi_{4n}(S^{2n+1}) \cong T_{2n}$ unless $n = 1,2,4$ when $\pi_{4n}(S^{2n+1}) = \mathbf{Z}/2r\mathbf{Z}$ with r as above. ∎

Having chosen once and for all a canonical generator ξ for the free component of $\pi_{4n-1}(S^{2n})$, we may freely speak of the "torsion component" of an element of $\pi_{4n-1}(S^{2n})$ or, more generally, of $\pi_{4n-1}(\bigvee S^{2n})$.

We now introduce a more convenient way of representing the elements of $\pi_{4n-1}(S^{2n})$.

It follows from Theorem 1.4 that $\Sigma(a\eta + x\omega) = (a - 2x)\Sigma(\eta)$ for $n = 1$, 2 or 4. Hence, in these cases,

$$2x(f)\Sigma(\eta) = a\Sigma(\eta) - \Sigma(f)$$

where we have written $x(f)$ for x to emphasise its dependence on f.

Thus f is determined up to homotopy by the pair $(H(f),\Sigma(f))$, since this pair uniquely determines $H(f)\eta + x(f)\omega$. Of course f also determines the pair $(H(f),\Sigma(f))$.

If $n \neq 1$, 2 or 4, then each function $f: S^{4n-1} \to S^{2n}$ is the form $a\xi + \zeta$, where $\xi = [\iota,\iota]$ and where ζ is now an element of the finite abelian group T_{2n}. But then $H(f) = a(H[\iota,\iota]) = 2a$, $\Sigma(\xi) = 0$ and the suspension morphism restricts to an isomorphism $T_{2n} \to \pi_{4n}(S^{2n+1})$, so that $\Sigma(f) = \Sigma(\zeta)$. Thus the pair $(H(f),\Sigma(f))$ once again determines f up to homotopy.

These considerations allow us to represent any function $f: S^{4n-1} \to S^{2n}$ (without restriction on n) as the pair $(H(f),\Sigma(f))$. Moreover, this representation is natural in the sense that it is compatible with the operations of $\pi_{4n-1}(S^{4n-1})$ and $\pi_{2n}(S^{2n})$ on $\pi_{4n-1}(S^{2n})$.

THEOREM 1.5. *For maps $\psi: S^{4n-1} \to S^{4n-1}$, $f: S^{4n-1} \to S^{2n}$ and $\varphi: S^{2n} \to S^{2n}$, we have*

$$(H(f\circ\psi),\Sigma(f\circ\psi)) = (\deg(\psi)H(f), \deg(\psi)\Sigma(f)),$$
$$(H(\varphi\circ f),\Sigma(\varphi\circ f)) = (\deg(\varphi)^2 H(f), \deg(\varphi)\Sigma(f)).$$

PROOF. The equalities are immediate consequences of Theorems 1.1 and 1.3. ∎

The last theorem also provides purely algebraic conditions equivalent to the mapping cones C_f and C_g of $f,g: S^{4n-1} \to S^{2n}$ being homotopically equivalent.

COROLLARY 1.6. *The mapping cones of $f,g: S^{4n-1} \to S^{2n}$ are homotopically equivalent if and only if*

$$H(g) = \pm H(f) \quad \text{and} \quad \Sigma(g) = \pm\Sigma(f).$$

PROOF. A self-map of S^m ($m \geq 1$) is a homotopy equivalence if and only if its degree is ± 1. ∎

In particular, when discussing the homotopy properties of the mapping cone C_f, we may assume that the Hopf invariant of f is non-negative.

2. Localisation and the genus of $S^{2n} \cup_f e^{4n}$

We denote by $X_{(p)}$ the *p-localisation* of the space X, by $e: X \to X_{(p)}$ the *p-localising map* and by $f_{(p)}$ the *p-localisation* of the map f in the sense of [BK], [HMR] or [HP].

The spaces we study in this paper are mapping cones of maps $f: S^{4n-1} \to S^{2n}$. Given $f, g: S^{4n-1} \to S^{2n}$, maps $(C_{(f)})_{(p)} \to (C_{(g)})_{(p)}$ correspond to commutative diagrams

$$\begin{array}{ccc} (S^{4n-1})_{(p)} & \xrightarrow{f_{(p)}} & (S^{2n})_{(p)} \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ (S^{4n-1})_{(p)} & \xrightarrow[g_{(p)}]{} & (S^{2n})_{(p)} \end{array}$$

The horizontal arrows are classified up to homotopy by $[(S^{4n-1})_{(p)}, (S^{2n})_{(p)}] \cong [S^{4n-1}, S^{2n}]_{(p)}$, which is, by Theorem 1.4, $\pi_{4n-1}(S^{2n})_{(p)} \cong \pi_{4n-1}(S^{2n}) \otimes \mathbf{Z}_{(p)}$. Thus $(a\xi + \zeta)_{(p)}$ is $(a\xi_{(p)} + \zeta_{(p)})$.

Similarly the vertical arrows are classified up to homotopy by $\pi_m(S^m)_{(p)} \cong \mathbf{Z}_{(p)}$. Hence we may write ψ' as r/s and φ' as t/u with s and u coprime to p. Now $\| \zeta_{(p)} \|$, the order of $\zeta_{(p)}$, is a power of p, say $\| \zeta_{(p)} \| = p^l$. Let $d = 2^{l+1}$ if $p = 2$ and $n = 2,4$. Otherwise let $d = p^l$. Then there are integers v and w such that $vsu + wd = 1$. Take self maps ρ of $(S^{4n-1})_{(p)}$ and σ of $(S^{2n})_{(p)}$ classified by vsu and $(vsu)^2$ respectively. Then, by Theorem 1.5, $\sigma \circ g_{(p)} \simeq g_{(p)} \circ \rho$ since $vsu \equiv 1$ $(\mathrm{mod} \| \zeta_{(p)} \|)$. Moreover, ρ and σ are homotopy equivalences—their homotopy inverses are classified by $1/(vsu)^2$ and $1/vsu$ respectively. Hence we may replace ψ' by $\psi := \rho \circ \psi'$ and φ' by $\varphi := \sigma \circ \varphi'$ in the diagram above. Furthermore, since they are classified by integers, ψ and φ are the localisations of self-maps of S^{4n-1} and S^{2n}, namely of maps of degree vru and vst respectively.

These considerations mean that we may carry out our "localised" computations in the "global" setting. The next lemma is a formal restatement of this fact.

LEMMA 2.1. *The mapping cones of $f, g: S^{4n-1} \to S^{2n}$ are p-equivalent if and only if there is a homotopy commutative diagram*

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{f} & S^{2n} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ S^{4n-1} & \xrightarrow[g]{} & S^{2n} \end{array}$$

with both $\deg(\psi)$ *and* $\deg(\varphi)$ *coprime to p. In other words, C_f and C_g are p-equivalent if and only if there are integers k and l coprime to p such that*

$$lH(g) = k^2 H(f) \quad \textit{and} \quad l\Sigma(g) = k\Sigma(f). \qquad \blacksquare$$

We can now classify the mapping cones of maps $f: S^{4n-1} \to S^{2n}$ into their respective genera.

THEOREM 2.2 (Classification Theorem). *The mapping cones C_f and C_g of the maps $f, g: S^{4n-1} \to S^{2n}$ are of the same genus if and only if*

(i) *the Hopf invariants of f and g agree up to sign, and*

(ii) *their suspensions are of the same genus.*

PROOF. Suppose first that C_f and C_g are of the same genus, and let p be a prime number. Then by assumption $(C_f)_{(p)} \simeq (C_g)_{(p)}$. Hence, by Lemma 2.1, there are maps $\psi: S^{4n-1} \to S^{4n-1}$ and $\varphi: S^{2n} \to S^{2n}$ of degrees l and k respectively, both coprime to p, such that

$$\begin{aligned}(lH(g), l\Sigma(g)) &= (H(g \circ \psi), \Sigma(g \circ \psi)) = (H(\varphi \circ f), \Sigma(\varphi \circ f)) \\ &= (k^2 H(f), k\Sigma(f)).\end{aligned}$$

It follows immediately that p^r divides $H(f)$ if and only if p^r divides $H(g)$ and that $\Sigma C_f = C_{\Sigma(f)}$ and $\Sigma C_g = C_{\Sigma(g)}$ are p-equivalent. This being true for every prime p, it follows (a) that $H(f)$ and $H(g)$ have the same prime factorisation, so that $H(g) = \pm H(f)$, and (b) that ΣC_f and ΣC_g are of the same genus.

By the remark after Corollary 1.6 we may suppose for the converse that $H(f) = H(g)$.

Now ΣC_f and ΣC_g are of the same genus if and only if $\Sigma(f)$ and $\Sigma(g)$ generate the same subgroup of $\pi_{4n}(S^{2n+1})$ [M]. Thus there is an integer k coprime to $\|T\|$, the order of the torsion subgroup T of $\pi_{4n-1}(S^{2n})$, such that $\Sigma C_f = k \cdot \Sigma C_g$. Choosing self-maps φ of S^{2n} and ψ of S^{4n-1} of degrees k and k^2 respectively, we have

$$\varphi \circ (H(f), \Sigma(f)) = (k^2 H(f), k\Sigma(f)) = (k^2 H(g), k^2 \Sigma(g)) = (H(g), \Sigma(g)) \circ \psi.$$

Hence C_f and C_g are p-equivalent for every prime p which divides the order of $\pi_{4n}(S^{2n})$. If, on the other hand, the prime p does not divide $\pi_{4n}(S^{2n})$, then $(x(f))_{(p)} = (x(g))_{(p)} = 0$ so that C_f and C_g p-localise to the same space.

Thus C_f and C_g are of the same genus. ■

As a corollary we have Theorem A from the Introduction.

COROLLARY 2.4. *The mapping cones C_f and C_g of $f, g: S^{4n-1} \to S^{2n}$ are of the same genus if and only if*

(i) *f and g generate the same subgroup of $\pi_{4n-1}(S^{2n})$ modulo torsion and*

(ii) *$\Sigma(f)$ and $\Sigma(g)$ generate the same subgroup of $\pi_{4n}(S^{2n+1})$.*

PROOF. Note that (i) is equivalent here to $H(g) = \pm H(f)$. ■

3. Examples

As an application of the Classification Theorem, we use Theorems 1.4 and 2.2 above to compute explicitly the genera of the complex projective plane **CP**(2), the quaternionic projective plane **HP**(2) and the Cayley "projective plane" **OP**(2). They are the mapping cones C_η of the respective Hopf maps $\eta : S^{4n-1} \to S^{2n}$, $n =$ 1, 2 and 4. In each case the torsion subgroup T of $\pi_{4n-1}(S^{2n})$ is cyclic generated by, say, ω of order t and $\pi_{4n}(S^{2n+1})$ is finite cyclic of order $2t$. The kernel of the suspension morphism is generated by $[\iota, \iota] = 2\eta + \omega$, the Whitehead square of the identity map id : $S^{2n} \to S^{2n}$.

Moreover, each η has Hopf invariant 1 and it follows that $(-1) \circ \eta = \eta + \omega$. Hence the mapping cone of $a\eta + x\omega$ is homotopy equivalent to that of $a\eta + y\omega$ where $x + y \equiv a \pmod t$. Since $t = 1$, 12 or 120 according as $n = 1$, 2 or 4, we may restrict attention in considering spaces of the genus of **CP**(2), **HP**(2) or **OP**(2) to mapping cones of maps of the form $\eta + 2k\omega$ with $k = 0, \ldots, 5$ in the case of **HP**(2) and $k = 0, \ldots, 59$ in the case of **OP**(2).

The Case $n = 1$. Since $\pi_3(S^2) \cong \mathbf{Z}$ is torsion-free, there is only one homotopy class of maps $f: S^3 \to S^2$ with Hopf invariant 1, so that $\mathbf{G}(\mathbf{CP}(2)) = \{C_\eta\}$.

The Case $n = 2$. The relevant homotopy group is $\pi_7(S^4) \cong \mathbf{Z} \oplus \mathbf{Z}/12\mathbf{Z}$. There are six distinct homotopy classes of mapping cones C_f with $H(f) = 1$, namely those of $\{\eta + 2k\omega \mid k = 0,1,2,3,4,5\}$. Their suspensions are the mapping cones $\Sigma(\eta)$, $21\Sigma(\eta)$, $17\Sigma(\eta)$, $13\Sigma(\eta)$, $9\Sigma(\eta)$, and $5\Sigma(\eta)$ respectively. These are of the same genus if and only if their attaching maps generate the same subgroup of $\mathbf{Z}/24\mathbf{Z}\Sigma(\eta)$. The six maps fall into two classes, namely those with $k \in \{0,2,3,5\}$ and those with $k \in \{1,4\}$, each class determining a complete genus. Thus $\mathbf{G}(\mathbf{HP}(2)) = \{C_\eta, C_{\eta+4\omega}, C_{\eta+6\omega}, C_{\eta+10\omega}\}$. The other genus of mapping cones C_f with $f \in \pi_7(S^4)$ and $H(f) = 1$ is $\{C_{n+2\omega}, C_{\eta+8\omega}\}$.

The Case $n = 4$. The relevant homotopy group is $\pi_{15}(S^8) \simeq \mathbf{Z} \oplus \mathbf{Z}/120\mathbf{Z}$, and there are 60 distinct homotopy types of mapping cones of maps $f: S^{15} \to S^8$ with the attaching map of Hopf invariant 1. There are four genera, with 32, 16, 8 and 4 distinct homotopy types respectively. The Cayley projective plane is in the largest genus, namely,

$$C_{\eta+kw} \in \mathbf{G}(\mathbf{OP}(2)) \qquad \text{if and only if } k = 2r \text{ with}$$

$$r \in \{0,2,3,5,6,8,11,12,15,17,18,19,20,21,23,26,27,$$

$$30,32,33,35,36,38,42,45,47,48,50,51,53,56,57\}.$$

We spare the reader the lists of the other genera and offer in their stead an observation about the genera in the three cases just considered.

The binary expansion of the number of homotopy types of mapping cones C_f with $H(f) = 1$ provides the partition of the homotopy types into genera: $n = 1$: There is only one homotopy type. $n = 2$: The $6 = 110_2$ homotopy types fall into 2 genera, one with 4 homotopy types, the other with 2. $n = 4$: The $60 = 111100_2$ fall into 4 genera, one with 32 homotopy types, one with 16, one with 8 and the last with 4.

4. Cancellation properties of $S^{2n} \cup_f e^{4n}$

We now turn to the $\vee$-cancellation properties of the mapping cones C_f of maps $f: S^{4n-1} \to S^{2n}$.

We saw in the last section that the genus of C_f can be characterised algebraically without any assumption on the order of $f \in \pi_{4n-1}(S^{2n})$. The cancellation behaviour of C_f on the other hand depends vitally on whether or not this order is finite.

Recall that the order of $f \in \pi_{4n-1}(S^{2n})$ is finite if and only if f is a suspension element. According to the results of [M], [CW] and [Z], the genus of C_f is then characterised by non-cancellation phenomena as well as by an algebraic condition.

THEOREM 4.1. *Given $f, g: S^{m-1} \to S^n$ of finite order in $\pi_{m-1}(S^n)$ $(m > n > 1)$, the following statements are equivalent*:

(i) *C_f and C_g are of the same genus.*

(ii) $C_f \vee S^n \vee S^m \simeq C_g \vee S^n \vee S^m$.

(iii) *$C_f \vee B \simeq C_g \vee B$ for some suitable bouquet of spheres B.*

If, additionally, f is a suspension element, then these are equivalent to:

(iv) *For some integer K,* $\bigvee_{k=1}^{K} C_f \simeq \bigvee_{k=1}^{K} C_g$. ■

Since the wedge of mapping cones is the mapping cone of the wedge of the attaching maps, our investigation of the cancellation properties reduces to the investigation of our fundamental diagram

$$\begin{array}{ccc} \bigvee S^{4n-1} & \xrightarrow{f} & \bigvee S^{2n} \\ \psi \downarrow & & \downarrow \varphi \\ \bigvee S^{4n-1} & \xrightarrow[g]{} & \bigvee S^{2n} \end{array}$$

We generalise from maps $S^{4n-1} \to S^{2n}$ to maps $\bigvee S^{4n-1} \to \bigvee S^{2n}$ and give an algebraic account of the homotopy classes of maps between the mapping cones of such functions, beginning with some notational conventions.

NOTATION. Denote by in_j the j-th canonical inclusion in the co-product $X_j \to \bigvee X_\iota$ and by $q_i: \bigvee X_\iota \to X_i$ the map which collapses the wedge onto the i-th summand.

It follows from general principles that $[\bigvee X_\iota, Y] \cong \prod [X_\iota, Y]$, so that we need only determine the structure and properties of $\pi_l(\bigvee S^m) := [S^l, \bigvee S^m]$. The Hilton–Milnor Theorem describes the structure of this homotopy group.

THEOREM 4.2 (Hilton–Milnor Theorem) [GWW]. *As abelian group*

$$\pi_k(S^m) \cong \bigoplus_{i=1}^{K} \pi_k(S^m) \oplus \bigoplus_{j\in\mathbf{N}} \pi_k(S^{m_j})$$

where $\{m_j \mid j \in \mathbf{N}\}$ *is a non-decreasing sequence of natural numbers of the form* $(s+1)m - s$ *with* $s \in \mathbf{N}\setminus\{0\}$.

If $m_j = (s+1)m - s$, *then the factor* $\pi_k(S^{m_j})$ *is imbedded by the induced morphism*

$$w_j(\iota_1, \ldots, \iota_K)_\# : \pi_k(S^{m_j}) \to \pi_k\left(\bigvee_{i=1}^{K} S^m\right)$$

where $w_j(\iota_1, \ldots, \iota_K)$ *is a suitable* $(s+1)$*-fold Whitehead product of the elements* $\iota_j := [in_j] \in \pi_m(\bigvee S^m)$. *The other embeddings are the morphisms*

$$(in_i)_\# : \pi_k(S^m) \to \pi_k(\vee S^m). \qquad \blacksquare$$

Of particular interest to us are the next two corollaries.

COROLLARY 4.3. *If* $k < 2m - 1$ *then*

$$\pi_k\left(\bigvee_{i=1}^{K} S^m\right) \cong \bigoplus_{i=1}^{K} \pi_k(S^m),$$

so that as abelian group

$$\left[\bigvee_{i=1}^{K} S^m, \bigvee_{i=1}^{K} S^m\right] \cong \mathbf{Z}^{K\times K}. \qquad \blacksquare$$

COROLLARY 4.4.

$$\pi_{2m-1}\left(\bigvee_{i=1}^{K} S^m\right) \cong \bigoplus_{i=1}^{K} \pi_{2m-1}(S^m) \oplus \bigoplus_{i=1}^{K(K-1)/2} \pi_{2m-1}(S^{2m-1}),$$

where the summands $\pi_{2m-1}(S^{2m-1})$ *are embedded by composition with the Whitehead products* $[\iota_i, \iota_j]$ $(i<j)$. *The other inclusions are* $(in_h)_\# : \pi_{2m-1}(S^m) \to \pi_{2m-1}(\bigvee_{k=1}^{K} S^m)$. $\blacksquare$

We may thus write each $\alpha \in \pi_{4n-1}(\bigvee_{i=1}^{K} S^{2n})$ as

$$\sum_{h=1}^{K} (a_{hh}\xi_h + \zeta_{hh}) + \sum_{1\leq i<j\leq K} a_{ij}[\iota_i, \iota_j]$$

where $q_h \circ \alpha = a_{hh}\xi_h + \zeta_{hh} = in_h \circ (a_{hh}\xi + \zeta_h)$.

We next extend the notation of Section 1, replacing the torsion component of α by the suspension $\Sigma\alpha$ and the torsion-free part by the $K \times K$ integral quadratic form $\underline{H}(\alpha)$ whose matrix (x_{ij}) is given by

$$x_{ii} = H(q_i \circ \alpha),$$

$$x_{ij} = x_{ji} = a_{ij} \qquad \text{if } i < j.$$

That is, the coefficients are the Hilton-Hopf invariants of α. Note that since

$$H(q_h \circ \alpha) = \begin{cases} a_{hh} & \text{if } n = 1, 2 \text{ or } 4, \\ 2a_{hh} & \text{otherwise,} \end{cases}$$

a_{hh} and $H(q_h \circ \alpha)$ determine each other uniquely.

We call $\underline{H}(\alpha)$ the *Hilton-Hopf quadratic form of* α. If $K = 1$, then this is equivalent to the classical Hopf invariant. We use underlining in our notation to emphasise that in the general case we are dealing with "matrices" (over a suitable group).

If $n \neq 1,2,4$, then the suspension map restricts to an isomorphism of the torsion component of $\pi_{4n-1}(\bigvee S^{2n})$ onto $\pi_{4n}(\bigvee S^{2n+1})$, so that $\underline{\zeta}(\alpha)$ and $\Sigma\alpha$ determine each other. If, on the other hand, $n = 1$, 2, or 4, then $\Sigma\alpha$ together with $\underline{H}(\alpha)$ determine $\underline{\zeta}(\alpha)$ and $\underline{H}(\alpha)$ uniquely, for the suspension restricts to a monomorphism of the torsion component of $\pi_{4n-1}(\bigvee S^{2n})$ into $\pi_{4n}(\bigvee S^{2n+1})$, even though it is no longer an isomorphism. Thus $\Sigma\alpha$ determines $\underline{\zeta}(\alpha)$ in the presence of $\underline{H}(\alpha)$. (The converse is obvious.) We regard $\Sigma\alpha$ as a column "K-vector" whose k-th entry is $\zeta_{kk} = in_k \circ \zeta_k$.

We now turn to determining $\underline{H}(\varphi \circ f)$ and $\Sigma(\varphi \circ f)$ for $f: S^{4n-1} \to \bigvee S^{2n}$ and $\varphi: \bigvee S^{2n} \to \bigvee S^{2n}$, beginning with an algebraic description of the structure of $[\bigvee S^m, \bigvee S^m]$ $(m > 1)$. Because of the central rôle they play, we summarise the properties of the inclsuions $in_j: S^m \to \bigvee S^m$ and the collapsing maps $q_i: \bigvee S^m \to S^m$.

LEMMA 4.5. *Denoting the Kronecker symbol by* δ_{ij},

(i) $q_i \circ in_j = \delta_{ij}\,\mathrm{id}_{S^m}: S^m \to S^m$.

(ii) $in_j \circ q_j: \bigvee S^m \to \bigvee S^m$ *is a non-zero idempotent which is non-trivial if* $K > 1$.

In fact, these maps induce a splitting of the identity map of $\pi_r(\bigvee S^m)$ *into a sum of idempotents*:

$$\mathrm{id} = \sum (in_k \circ q_k)_\#: \pi_r(\bigvee S^m) \to \pi_r(\bigvee S^m).$$

(iii) *The function* $\underline{A}: [\bigvee S^m, \bigvee S^m] \to \mathbf{M}(K;\mathbf{Z})$ *mapping* φ *to the* $K \times K$ *integral matrix* $\underline{A}(\varphi)$, *whose* (i,j)*-th coefficient is the degree of* $\varphi_{ij} := q_i \circ \varphi \circ in_j$, *is a natural isomorphism of rings.* ∎

Corollary 4.6. *The self-map* $\varphi: \bigvee S^m \to \bigvee S^m$ *is*
(i) *a homotopy equivalence if and only if* $\det(\underline{A}(\varphi)) = \pm 1$ *and*
(ii) *a p-equivalence if and only if* $\det(\underline{A}(\varphi))$ *is coprime to p.* ∎

A final corollary provides the model for many later computations.

Corollary 4.7. *Given* $\varphi: \bigvee S^m \to \bigvee S^m$, $\varphi \circ in_j = \sum_{i=1}^K \deg(\varphi_{ij}) in_i$.

Proof. It follows from Lemma 4.5 that $\varphi \circ in_j = \left(\sum_{i=1}^K (in_i \circ q_i)\right) \circ \varphi \circ in_j$. By the Hilton–Milnor Theorem, composition is left-additive in this range, so that $\varphi \circ in_j = \sum_{i=1}^K in_i \circ q_i \circ \varphi \circ in_j$. By definition $q_i \circ \varphi \circ in_j = \varphi_{ij}$ and composition is right-additive. Hence

$$\varphi \circ in_j = \sum_{i=1}^K in_i \circ \varphi_{ij} = \sum_{i=1}^K \deg(\varphi_{ij}) in_i. \qquad ∎$$

Lemma 4.8. *Given* $f: S^{4n-1} \to \bigvee S^{2n}$ *and* $\varphi: \bigvee S^{2n} \to \bigvee S^{2n}$,

$$\underline{H}(\varphi \circ f) = \underline{A}(\varphi)\underline{H}(f)(\underline{A}(\varphi))^t,$$

$$\Sigma(\varphi \circ f) = \underline{A}(\varphi)\Sigma f.$$

Proof. Computations in the style of Corollary 4.7 establish the result. We spare the reader the debauch of indices. ∎

The corresponding computations for $g \circ \psi$ are much simpler, since composition is always right-additive.

Lemma 4.9. *Given* $\psi: \bigvee S^{2n} \to \bigvee S^{2n}$ *and* $g: S^{4n-1} \to \bigvee S^{2n}$,

$$\underline{H}(g \circ \psi) = \deg(\psi)\underline{H}(g),$$

$$\Sigma(g \circ \psi) = \deg(\psi)\Sigma g. \qquad ∎$$

The last two results generalise Theorem 1.5. The next proposition summarises them in a form convenient for calculations.

Proposition 4.10. *The maps* $\varphi \circ f$, $g \circ \psi: S^{4n-1} \to \bigvee S^{2n}$ *are homotopic if and only if*

$$\deg(\psi)\underline{H}(g) = \underline{A}(\varphi)\underline{H}(f)(\underline{A}(\varphi))^t$$

and

$$\deg(\psi)\Sigma g = \underline{A}(\varphi)\Sigma f. \qquad \blacksquare$$

We now return to the task of algebraically characterising our fundamental diagram

$$\begin{array}{ccc} \bigvee S^{4n-1} & \xrightarrow{f} & \bigvee S^{2n} \\ \psi \downarrow & & \downarrow \varphi \\ \bigvee S^{4n-1} & \xrightarrow[g]{} & \bigvee S^{2n} \end{array}$$

The horizontal maps are classified by $[\bigvee S^{4n-1}, \bigvee S^{2n}] \cong \bigoplus \pi_{4n-1}(\bigvee S^{2n})$ with as many direct summands as there are $(4n-1)$-spheres in the bouquet $\bigvee S^{4n-1}$. Thus we have

$$[\bigvee S^{4n-1}, \bigvee S^{2n}] \overset{\cong}{\to} \bigoplus \pi_{4n-1}(\bigvee S^{2n}), \quad \alpha \mapsto (\alpha^1, \ldots, \alpha^K)$$

where $\alpha^i := \alpha \circ in_i \in \pi_{4n-1}(\bigvee S^{2n})$ $(i = 1, \ldots, K)$, and we can also write

$$\alpha = \sum_{k=1}^{K} \alpha^k.$$

Furthermore, given $\psi : \bigvee S^{4n-1} \to \bigvee S^{4n-1}$, we use Lemma 4.5 and Corollary 4.7 to find that

$$(\alpha \circ \psi)^k = \sum_{i=1}^{K} \deg(\psi_{ik})\alpha^i.$$

So, writing $\underline{V}(\alpha)$ for $(\alpha^1, \ldots, \alpha^K)$, and with $\underline{A}(\psi)$ defined as above, we have established the following proposition.

PROPOSITION 4.11. *Given* $\psi : \bigvee S^{4n-1} \to \bigvee S^{4n-1}$ *and* $\alpha \in [\bigvee S^{4n-1}, \bigvee S^{2n}]$,

$$\underline{V}(g \circ \psi) = \underline{V}(g) \cdot \underline{A}(\psi). \qquad \blacksquare$$

We can represent each $\alpha \in [\bigvee_{j=1}^{J} S^{4n-1}, \bigvee_{k=1}^{K} S^{2n}]$ as a pair $(\underline{H}(\alpha), \Sigma(\alpha))$, where $\underline{H}(\alpha)$ is the $K \times JK$ integral matrix given by juxtaposing the J symmetric integral matrices $\underline{H}(\alpha^j)(j = 1, \ldots, J)$ and $\Sigma(\alpha)$; the suspension of α can be thought of as a $K \times J$ "matrix" over $\pi_{4n}(S^{2n+1})$. Thus the $(i, (k-1)K + j)$-th coefficient of $\underline{H}(\alpha)$ is the (i,j)-th coefficient of $\underline{H}(\alpha^k)$ and the i-th column of $\Sigma(\alpha)$ is the suspension of α^i. In fact the torsion component $\underline{\zeta}(\alpha)$ of α and $\Sigma(\alpha)$ determine each other: we apply our earlier argument column by column.

We extend the matrix notation to this new situation, using the Kronecker product of matrices.

DEFINITION. Given an $I \times J$ matrix $A = (a_{ij})$ and a $K \times L$ matrix $B = (b_{kl})$, their *Kronecker product* $A \otimes B$ is the matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1J}B \\ a_{21}B & a_{22}B & \cdots & a_{2J}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{I1}B & a_{I2}B & \cdots & a_{IJ}B \end{pmatrix}$$

In other words $A \otimes B$ has $a_{ij}b_{kl}$ as the (k,l)-th coefficient in the (i,j)-th block.

COROLLARY 4.12. *Given* $\psi: \bigvee S^{4n-1} \to \bigvee S^{4n-1}$, $\varphi: \bigvee S^{2n} \to \bigvee S^{2n}$ *and* $\alpha \in [\bigvee S^{4n-1}, \bigvee S^{2n}]$,

$$\underline{H}(\varphi \circ \alpha \circ \psi) = \underline{A}(\varphi)\underline{H}(\alpha)\left(\underline{A}(\psi) \otimes \underline{A}(\varphi)^t\right),$$

$$\Sigma(\varphi \circ \alpha \circ \psi) = \underline{A}(\varphi)\Sigma(\alpha)\underline{A}(\psi). \qquad \blacksquare$$

We can now prove our Factorisation Theorem (Theorem D of the introduction).

THEOREM 4.13 (Factorisation Theorem). *Suppose* $f_i, g_i: S^{4n-1} \to S^{2n}$ $(i = 1, \ldots, K)$ *are given and that the Hopf invariants* $H(f_i)$ *and* $H(g_j)$ *are non-zero if and only if* $i \leq L$ *and* $j \leq M$. *Then*

$$\bigvee_{j=1}^{K} C_{f_i}, \text{ is homotopy equivlaent to } \bigvee_{k=1}^{K} C_{g_j}$$

if and only if

(i) $L = M$,

(ii) C_{f_i} *is homotopy equivalent to* $C_{g_{\sigma(i)}}$ *for some permutation* σ *of* $\{1, \ldots, L\}$,

(iii) $C_{f_{L+1}} \vee \cdots \vee C_{f_K}$ *is homotopy equivalent to* $C_{g_{L+1}} \vee \cdots \vee C_{g_K}$.

PROOF. Clearly, only the "only if" part requires proof. So suppose that

$$\bigvee_{j=1}^{K} C_{f_i} \simeq \bigvee_{k=1}^{K} C_{g_j}.$$

Defining $f, g: \bigvee S^{4n-1} \to \bigvee S^{2n}$ by $f := \bigvee_{j=1}^{K} f_i$ and $g := \bigvee_{j=1}^{K} g_i$, we have

$$\underline{H}(g)\left(\underline{A}(\psi) \otimes \underline{1}_K\right) = \underline{A}(\varphi)\underline{H}(f)\left(\underline{1}_K \otimes \underline{A}(\varphi)^t\right)$$

and

$$\Sigma g \underline{A}(\psi) = \underline{A}(\varphi)\Sigma f,$$

with $\det \underline{A}(\psi), \det \underline{A}(\varphi) = \pm 1$. We see that $f^i = in_i \circ f_i$, so that $\underline{H}(f^i)$ has at most one non-zero entry, $H(f_i)$ in the i-th position on the main diagonal. Thus the rank of $\underline{H}(f)$ is precisely L, the number of f_i's whose Hopf invariant is non-zero. Of course $\underline{H}(g)$ has a similar form and its rank is M. Since both $\underline{A}(\psi)$ and $\underline{A}(\varphi)$ are invertible, it follows that $\underline{H}(f)$ and $\underline{H}(g)$ have the same rank, i.e. $L = M$, proving (i).

Comparing the k-th block of each of the matrices, we see that

$$\begin{bmatrix} \deg(\psi_{1k})H(g_1) & & 0 \\ & \ddots & \\ 0 & & \deg(\psi_{Kk})H(g_K) \end{bmatrix}$$

$$= H(f_k)\begin{bmatrix} \deg(\varphi_{1k})^2 & \cdots & \deg(\varphi_{1k})\deg(\varphi_{Kk}) \\ \vdots & \ddots & \vdots \\ \deg(\varphi_{Kk})\deg(\varphi_{1k}) & \cdots & \deg(\varphi_{Kk})^2 \end{bmatrix}$$

Choose $k \leq L$. Then $H(f_k) \neq 0$. Since $\underline{A}(\varphi)$ is invertible there is a $\sigma(k)$ with $\deg(\varphi_{\sigma(k)k}) \neq 0$. Then $\deg(\psi_{\sigma(k)k}), H(g_{\sigma(k)}) \neq 0$, since $\deg(\psi_{\sigma(k)k})H(g_{\sigma(k)}) = \deg(\varphi_{\sigma(k)k})^2 H(f_k) \neq 0$.

Comparing the matrix coefficients off the principal diagonal, we see that $H(f_k)\deg(\varphi_{\sigma(k)k})\deg(\varphi_{sk}) = 0$ whenever $s \neq \sigma(k)$. Thus $\deg(\varphi_{\sigma(k)k})$ is the only non-zero entry in the k-th column of $\underline{A}(\varphi)$. Hence $\sigma: \{1, \ldots, L\} \to \{1, \ldots, L\}$ must be injective, since $\underline{A}(\varphi)$ has rank K.

Comparing the $\sigma(k)$-th row of $\underline{H}(g \circ \psi)$ with that of $\underline{H}(\varphi \circ f)$ we see that $\deg(\psi_{\sigma(k)s})H(g_{\sigma(k)}) = H(f_s)\deg(\varphi_{\sigma(k)s})^2$ from which it follows that $\deg(\psi_{\sigma(k)s}) = 0$ unless $s = k$. Hence $\deg(\psi_{(\sigma(k)k})$ is the only non-zero entry in the $\sigma(k)$-th row of $\underline{A}(\psi)$ and $\deg(\varphi_{\sigma(k)k})$ is the only non-zero entry in the $\sigma(k)$-th column of $\underline{A}(\varphi)$. Moreover both have absolute value 1, since $\underline{A}(\psi)$ and $\underline{A}(\varphi)$ are invertible. Thus $H(g_{\sigma(k)}) = \pm H(f_k)$.

For the suspensions, we have the "matrix" equation $\Sigma g \underline{A}(\psi) = \underline{A}(\varphi)\Sigma f$, or

$$\begin{bmatrix} \Sigma g_1 & & 0 \\ & \ddots & \\ 0 & & \Sigma g_K \end{bmatrix}\begin{bmatrix} \deg(\psi_{11}) & \cdots & \deg(\psi_{1K}) \\ \vdots & & \vdots \\ \deg(\psi_{K1}) & \cdots & \deg(\psi_{KK}) \end{bmatrix}$$

$$= \begin{bmatrix} \deg(\varphi_{11}) & \cdots & \deg(\varphi_{1K}) \\ \vdots & & \vdots \\ \deg(\varphi_{K1}) & \cdots & \deg(\varphi_{KK}) \end{bmatrix}\begin{bmatrix} \Sigma f_1 & & 0 \\ & \ddots & \\ 0 & & \Sigma f_K \end{bmatrix}$$

Comparing the $(\sigma(k),k)$-th entries, we see that $\deg(\psi_{\sigma(k)k})\Sigma g_{\sigma(k)} = \deg(\varphi_{\sigma(k)k})\Sigma f_k$, that is $\Sigma g_{\sigma(k)} = \pm\Sigma f_k$. Thus, by Corollary 1.6, C_{f_k} and $C_{g_{\sigma(k)}}$ are homotopy equivalent, proving (ii).

Finally, it follows from the above considerations that both

$$\begin{bmatrix} \deg(\psi_{(L+1)(L+1)}) & \cdots & \deg(\psi_{(L+1)K}) \\ \vdots & & \vdots \\ \deg(\psi_{K(L+1)}) & \cdots & \deg(\psi_{KK}) \end{bmatrix}$$

and

$$\begin{bmatrix} \deg(\varphi_{(L+1)(L+1)}) & \cdots & \deg(\varphi_{(L+1)K}) \\ \vdots & & \vdots \\ \deg(\varphi_{K(L+1)}) & \cdots & \deg(\varphi_{KK}) \end{bmatrix}$$

are invertible. Thus defining $f' := f_{L+1} \vee \cdots \vee f_K$, $g' := g_{L+1} \vee \cdots \vee g_K$ and choosing ψ' and φ' with the above matrices as $\underline{A}(\psi')$ and $\underline{A}(\varphi')$ respectively, we see that $\varphi' \circ f' \simeq g' \circ \psi'$ so that

$$C_{f_{L+1}} \vee \cdots \vee C_{f_K} \simeq C_{f'} \simeq C_{g'} \simeq C_{g_{L+1}} \vee \cdots \vee C_{gK}. \qquad \blacksquare$$

Theorems B and C from the Introduction are special cases of this theorem.

Corollary 4.14 (Theorem B). *If $f,g : S^{4n-1} \to S^{2n}$ are both of infinite order in $\pi_{4n-1}(S^{2n})$, then*

$$C_f \vee S^{2n} \vee S^{4n} \simeq C_g \vee S^{2n} \vee S^{4n} \qquad \text{if and only if } C_f \simeq C_g. \qquad \blacksquare$$

Corollary 4.15 (Theorem C). *If $f,g : S^{4n-1} \to S^{2n}$ are both of infinite order in $\pi_{4n-1}(S^{2n})$, then*

$$\bigvee_{i=1}^{K} C_f \simeq \bigvee_{i=1}^{K} C_g \qquad \text{if and only if } C_f \simeq C_g. \qquad \blacksquare$$

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